Instructions:

- Do not open the exam until you are instructed to do so.
- Write your name on the front of the exam.
- Answer the questions in the spaces provided. If you run out of room for an answer, continue on the back of the page.
- If you need more paper, ask one of the proctors and we will provide it.

Math 100A - Midterm - 11/1/2017

Name: ______________________________

Question:	1	2	3	4	5	Total
Points:	5	8	5	6	6	30
Score:						

1. Let G and H be groups.

   (a) (1 point) Give the definition of a *subgroup of* G.

   (b) (1 point) Let $\varphi\colon G \to H$ be a group homomorphism, give the definition of $\operatorname{Ker}(\varphi)$, the *kernel of* φ.

   (c) (3 points) Prove that the kernel of φ is a subgroup of G.

2. Let G be a **finite** group.

   (a) (4 points) Let $e \in G$ be the identity, and let $x \in G$ an arbitrary element. Prove that there exists a positive integer $n > 0$ such that $x^n = e$.

   (b) (4 points) Suppose $x^2 = e$ for all $x \in G$. Prove that G is abelian.

3. (a) (1 point) Let S be a set. Give the definition of an *equivalence relation* $\sim$ *on* S.

(b) (4 points) Let H be a subgroup of a group G. Prove that the relation

$$x \sim y \iff x * y^{-1} \in H$$

is an equivalence relation on G.

4. (a) (1 point) Let n be a positive integer. Define what it means for two integers $a, b \in \mathbb{Z}$ to be *congruent modulo n*.

(b) (5 points) Recall that

$$U_n := \{[a] \in \mathbb{Z}_n | (a, n) = 1\}.$$

Prove that if $[a] \in U_n$, then there exists $[b] \in U_n$ such that $[a \cdot b] = 1$ (in other words, multiplication in U_n has inverses).

5. Let G be a group.

(a) (1 point) Give the definition of $Z(G)$, the *center of* G.

(b) (5 points) Prove that the center of D_8 is $\langle r^2 \rangle = \{1, r^2\} \leq D_8$.